

TCHEBYCHEV'S CHARACTERISTIC OF REARRANGEMENT INVARIANT SPACE.

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ABSTRACT.

We introduce and investigate in this short article a new characteristic of rearrangement invariant (r.i.) (symmetric) space, namely so-called Tchebychev's characteristic.

We reveal an important class of the r.i. spaces - so called regular r. i. spaces and show that the majority of known r.i. spaces: Lebesgue-Riesz, Grand Lebesgue Spaces, Orlicz, Lorentz and Marcinkiewicz r.i. spaces are regular. But we construct after several examples of r.i. spaces without the regular property.

Applications - Probability theory and Statistics.

Key words and phrases: Rearrangement invariant (r.i.) space, regular r.i. space, Tchebychev's characteristic, fundamental function, Grand Lebesgue Space (GLS), measure, resonant, Probability, distribution, tail function, partial order, associate and conjugate (dual) space, relation of equivalence, Orlicz, Lorentz and Marcinkiewicz spaces, upper and lower estimates.

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1 Notations. Statement of problem.

Let $(\Omega, \mathcal{A}, \mu)$ be measure space with sigma-finite non-trivial measure μ and $(F, \|\cdot\| = \|\cdot\|_F)$ be any rearrangement invariant (r.i.) space over $(\Omega, \mathcal{A}, \mu)$.

The detail investigate of r.i. spaces see in the classical books [1], [8].

Hereafter C, C_j will denote any non-essential finite positive constants. As usually, for the measurable function $f : \Omega \rightarrow R$,

$$|f|_p(\Omega, \mu) = |f|_p(\mu) = |f|_p = \left[\int_{\Omega} |f(\omega)|^p \mu(d\omega) \right]^{1/p}, \quad 1 \leq p < \infty,$$

$L_p(\Omega, \mu) = L_p(\mu) = \{f : |f|_p < \infty\}$; m will denote usually Lebesgue measure, and we will write in this case $m(dx) = dx$; $|f|_{\infty} \stackrel{def}{=} \operatorname{vraisup}_{\omega} |f(\omega)|$.

We will conclude without loss of generality in the case when $\mu(\Omega) < \infty \Rightarrow \mu(\Omega) = 1$, call: "probabilistic case" and denote $\mathbf{P} = \mu$,

$$\mathbf{E}\xi = \int_{\Omega} \xi(\omega) \mathbf{P}(d\omega).$$

We presume for example construction that the source measurable space $(\Omega, \mathcal{A}, \mu)$ is sufficiently rich; it is suffices to set $\Omega = [0, 1]$ or $\Omega = [0, \infty)$ with Lebesgue measure m .

We denote as usually for arbitrary finite a.e.measurable function (random variable) $\xi(\omega)$ its *Tail function* $T_{\xi}(t)$ as follows:

$$T_\xi(t) = \mu\{\omega : |\xi(\omega)| \geq t\}, \quad t > 0.$$

The left inverse function to the $T_\xi(t)$ is denoted $\xi^*(t)$.

Definition 1.1. Tchebychev's characteristic $T_F(t), t > 0$ of the space $(F, \|\cdot\| = \|\cdot\|_F)$ is defined as follows:

$$T^{(F)}(t) = T^{(F, \|\cdot\|)}(t) \stackrel{def}{=} \sup_{\xi: \xi \in F, \|\xi\|_F=1} T_\xi(t). \quad (1.1)$$

Our aim is to investigate the function $T_F(t)$ for sufficiently greatest values $t : t > t_0 = \text{const} > 0$ for different classes of r.i. spaces $(F, \|\cdot\| = \|\cdot\|_F)$.

A possible applications of tail estimates: Functional Analysis, see the classical books of C.Bennet and R.Sharpley [1], S.G. Krein Yu.V. Petunin and E.M. Semenov [8], and also in the articles [16], [17], [18]; Probability Theory [19], [21]; Numerical Methods Monte-Carlo [20]; Statistics [10], [11], [12], theory of random processes and fields [7], [13] etc.

For instance, let θ_n be $w(n)$, $w(n) \rightarrow \infty$ at $n \rightarrow \infty$, n is volume of sample, consistent statistical estimate of an unknown parameter θ for which

$$\|w(n)|\theta_n - \theta|\|_F \leq \sigma.$$

We can construct the confidence interval for the value θ by means of inequality

$$\mathbf{P}(w(n)|\theta_n - \theta| \geq u) \leq T^{(F)}(u/\sigma).$$

2 Simple properties of Tchebychev's characteristic. Examples.

A. Note that

$$T^{(F)}(t) = \sup_{\xi: \xi \in F, \|\xi\|_F \leq 1} T_\xi(t). \quad (2.1)$$

Moreover,

$$\sup_{\xi: \xi \in F, \|\xi\|_F = C} T_\xi(t) = \sup_{\xi: \xi \in F, \|\xi\|_F \leq C} T_\xi(t) = T^{(F)}(t/C), \quad C = \text{const} > 0. \quad (2.2)$$

B. Let on the space $(F, \|\cdot\|)$ be an other norm $|||\cdot|||$ for which

$$|||\xi||| \geq C_1 \|\xi\|, \quad 0 < C_1 = \text{const} < \infty.$$

Then

$$T^{(F, \|\cdot\|)}(t) \leq T^{(F, |||\cdot|||)}(t/C_1). \quad (2.3)$$

Analogously, if

$$\|\xi\| \leq C_2 \|\|\xi\|\|, \quad 0 < C_2 = \text{const} < \infty,$$

then

$$T^{(F, \|\cdot\|)}(t) \geq T^{(F, \|\|\cdot\|\|)}(t/C_2). \quad (2.4)$$

C. Definition 2.1. A two tail functions $T_1(t)$ and $T_2(t)$ are equivalent, write: $T_1(\cdot) \sim T_2(\cdot)$ iff there exist three finite positive constants $t_0 > 0$, $0 < C_1 \leq C_2 < \infty$ for which

$$T_2(t/C_2) \leq T_1(t) \leq T_2(t/C_1), \quad t \geq t_0. \quad (2.5)$$

We will write also $T_1(\cdot) << T_2(\cdot)$, iff

$$T_1(t) \leq T_2(t/C_1), \quad t \geq t_0. \quad (2.6)$$

Evidently, the relation " $<<$ " is partial order on the set of all tail functions and the relation " \sim " is the relation of equivalence. Also if $T_1 << T_2$ and $T_2 << T_1$, then $T_1 \sim T_2$.

Corollary 2.1. If two norms on the space F $\|\cdot\|$ and $\|\|\cdot\|\|$ are equivalent in the usually sense, then

$$T^{(F, \|\cdot\|)}(\cdot) \sim T^{(F, \|\|\cdot\|\|)}(\cdot). \quad (2.7)$$

As we will see further, the converse proposition is'nt true.

Recall that the measure space is said to be resonant, if it is non-atomic or conversely completely atomic with all the atoms having equal measure, see [1], chapter 2, section 7.

D. Theorem 2.1. Let the measure $\mu(\cdot)$ be resonant; then for any r.i. space $(F, \|\cdot\|)$

$$T^{(F, \|\cdot\|)}(t) \leq \frac{C_3(F)}{t}. \quad (2.8)$$

Proof. It is known, see [1], chapter 2, section 2, theorem 2.2 that in the considered case

$$\|\xi\|F \geq C_4|\xi|_1.$$

We use further Tchebychev's inequality:

$$T^{(L_1)}(t) \leq C_5/t, \quad t > 0.$$

The assertion of the theorem 2.1 follows from the inequality (2.3).

E. Examples. 1. Classical Lebesgue-Riesz spaces.

We have in the case $\mu(\Omega) = \infty$ and $1 \leq p < \infty$:

$$T^{(L_p)}(t) = t^{-p}, \quad t > t_0.$$

When $\mu(\Omega) = 1$

$$T^{(L_p)}(t) = \min(1, t^{-p}), \quad t > t_0. \quad (2.9)$$

Indeed, the upper estimate it follows from Tchebychev's inequality; the lower estimate follows from the consideration of following example:

$$\mathbf{P}(\xi = t) = t^{-p}; \quad \mathbf{P}(\xi = 0) = 1 - t^{-p}, \quad t > 1.$$

Obviously,

$$T^{(L_\infty)}(t) = 0, \quad t > 1.$$

2. Generalized Lorentz space.

Let $\mu = \mathbf{P}$ and let $w = w(t)$, $t > 0$ be positive continuous strictly increasing function, $\lim_{t \rightarrow \infty} w(t) = \infty$. A generalized Lorentz space $L^{(w)}$ consists by definition on all the measurable functions $\xi(\omega)$ with finite norm (more precisely, quasinorm)

$$\|\xi\|_{L^{(w)}} = \sup_{t>0} [w(t) T_\xi(t)]. \quad (2.10)$$

We conclude as before

$$T^{L^{(w)}}(t) = \min(1, 1/w(t)). \quad (2.11)$$

Remark 2.1. We observe if $w(t) = t^p$, then $T^{L^{(w)}}(t) = T^{L_p}(t)$, $t > 1$, but the spaces $L^{(w)}$ and L_p are not isomorphic.

3 Tchebychev's characteristic and fundamental functions. Regular r.i. spaces.

We study in this section the relations between Tchebychev's characteristic and fundamental functions.

We impose on the measure μ here the restriction that it is diffuse: for arbitrary measurable set B there is its measurable subset D such that

$$\mu(D) = \mu(B)/2.$$

Recall that a fundamental function $\phi_F(\delta)$, $\delta \in (0, \infty)$ of the r.i. space $(F, \|\cdot\|)$ over the measurable space $(\Omega, \mathcal{A}, \mu)$ may be defined as follows:

$$\phi_F(\delta) \stackrel{def}{=} \sup_{D: \mu(D) \leq \delta} \|I(D)\|_F. \quad (3.1)$$

Here and further $I(D) = I(D, \omega)$ is an indicator function of the measurable set D .

The application of fundamental function in the functional analysis, in particular, in the theory of interpolation of operators is described in [1], [8]; the application in the theory of approximation see in [15].

Many examples of fundamental functions for different r.i. spaces are computed in the books [1], [8]. For the so-called Grand Lebesgue Spaces the fundamental functions are investigated and calculated in [9], [16].

Let us consider for instance the case of Orlicz's space $Or(N)$ over our measurable space, in which we assume the measure μ to be diffuse. We suppose also that the Young function $N = N(u)$ is in addition strictly monotonic on the positive semi-axes and continuous.

We will use in this article the Luxemburg norm in the space $Or(N)$:

$$\|\xi\|_{Or(N)} = \inf \left\{ k > 0, \int_{\Omega} N \left(\frac{|\xi(\omega)|}{k} \right) \mu(d\omega) \leq 1 \right\}. \quad (3.2)$$

The fundamental function of this space has a view

$$\phi_{Or(N)}(\delta) = \frac{1}{N^{-1}(1/\delta)}. \quad (3.3)$$

Hereafter $g^{-1}(t)$ will denote the inverse function to the function $g(\cdot)$.

Further, let $\xi \geq 0$, $\|\xi\|_{Or(N)} = 1$. Since the Young function $N = N(u)$ is strictly monotonic and continuous

$$\int_{\Omega} N(\xi(\omega)) \mu(d\omega) = 1,$$

therefore

$$T_{\xi}(t) \leq 1/N(t).$$

We conclude analogously to the Lebesgue-Riesz and Lorentz spaces considering the example

$$\mathbf{P}(\xi_0 = t) = 1/N(t) = 1 - \mathbf{P}(\xi_0 = 0), \quad t = \text{const} : N(t) > 1,$$

for which

$$\mathbf{E}N(\xi_0) = 1,$$

that

$$T^{(Or(N))}(t) = 1/N(t), \quad t > t_0. \quad (3.4)$$

Definition 3.1. The r.i. space $(F, \|\cdot\|_F)$ is said to be *regular* r.i. space, if

$$\left[\frac{1}{\phi_F(1/t)} \right]^{-1} = \frac{1}{T^{(F)}(t)}, \quad t > t_0. \quad (3.5)$$

The r.i. space $(F, \|\cdot\|_F)$ is said to be *weak regular* r.i. space, if

$$\left[\frac{1}{\phi_F(1/t)} \right]^{-1} \asymp \frac{1}{T^{(F)}(t)}, \quad t > t_0. \quad (3.6)$$

We have proved the following fact.

Theorem 3.1. The Orlicz's space $Or(N)$ over our measurable space, in which we assume the measure μ to be diffuse and suppose also that the Young function $N = N(u)$ is in addition strictly monotonic increase and continuous, is regular r.i. space.

If we replace the Luxemburg norm on some equivalent, we obtain the weak regular space.

Examples 3.1. For the spaces L_p over diffuse sigma-finite measure we have

$$\phi_{L_p}(\delta) = \delta^{1/p}, \quad T^{(L_p)}(1/\delta) = \delta^p = [\phi_{L_p}(\delta)]^{-1}.$$

Another examples of weak regular r.i. spaces are the classical Lorentz and Marcinkiewicz spaces.

4 Tchebychev's characteristic of associate regular r.i. spaces.

Recall that the associate r.i. space $(F', \|\cdot\|_{F'})$ to the space $(F, \|\cdot\|_F)$ consists on all the measurable functions $g : \Omega \rightarrow R$ with finite norm

$$\|g\|_{F'} = \sup_{\xi: \|\xi\|_F=1} \left| \int_{\Omega} g(\omega) \xi(\omega) \mu(d\omega) \right|. \quad (4.0)$$

Under some additional conditions (absolutely continuous norm etc.) the associate space may coincides with conjugate (dual) space $(F^*, \|\cdot\|_{F^*})$; for instance, it is true for Orlicz's space $(\Omega, N(u))$ iff the Young function $N(u)$ satisfies the Δ_2 condition.

Theorem 4.1. Assume again that $(\Omega, \mathcal{A}, \mu)$ is resonant measure space. Suppose also both the r.i. spaces $(F, \|\cdot\|_F)$, $(F', \|\cdot\|_{F'})$ are regular. Then

$$\left[\frac{1}{T(F)} \right]^{-1}(t) \cdot \left[\frac{1}{T(F')} \right]^{-1}(t) = t, \quad t > 0. \quad (4.1)$$

Proof. Since both the r.i. spaces $(F, \|\cdot\|_F)$, $(F', \|\cdot\|_{F'})$ are regular,

$$\phi_F(\delta) = \left[\frac{1}{T(F)} \right]^{-1} \left(\frac{1}{\delta} \right), \quad \phi_{F'}(\delta) = \left[\frac{1}{T(F')} \right]^{-1} \left(\frac{1}{\delta} \right). \quad (4.2)$$

We will use the known identity [1], chapter 2, section 5:

$$\phi_F(\delta) \cdot \phi_{F'}(\delta) = \delta. \quad (4.3)$$

It remains to substitute in equality (4.3) expressions (4.2) and write t instead $1/\delta$.

Corollary 4.1. If $F' = F^*$, then

$$\left[\frac{1}{T(F)} \right]^{-1}(t) \cdot \left[\frac{1}{T(F^*)} \right]^{-1}(t) = t, \quad t > 0. \quad (4.4)$$

Corollary 4.2. If both the r.i. spaces $(F, \|\cdot\|_F)$, $(F', \|\cdot\|_{F'})$ are weakly regular, then

$$\left[\frac{1}{T(F)} \right]^{-1}(t) \cdot \left[\frac{1}{T(F')} \right]^{-1}(t) \asymp t, \quad t > 0. \quad (4.5)$$

Corollary 4.3. The condition of theorem 4.1 is satisfied if for example the space F is Orlicz space with continuous strictly increasing Young function $N = N(u)$, $u \geq 0$.

Corollary 4.4. Without the condition of resonance we can guarantee only the inequality

$$\left[\frac{1}{T^{(F)}} \right]^{-1}(t) \cdot \left[\frac{1}{T^{(F')}} \right]^{-1}(t) \geq t, \quad t > 0. \quad (4.6)$$

This fact follows immediately from the inequality

$$\phi_F(\delta) \cdot \phi_{F'}(\delta) \geq \delta, \quad (4.7)$$

see also [1], chapter 2, section 5.

5 Tchebychev's characteristic of the direct sum of r.i. spaces.

Definition 5.1. We define for two tail functions $T_1(\cdot)$, $T_2(\cdot)$ the following operation:

$$T_1 \vee T_2(t) \stackrel{def}{=} \inf_{x \in [0,1]} [T_1(tx) + T_2(t(1-x))]. \quad (5.0)$$

Evidently, $T_1 \vee T_2(t)$ is again the tail function and $T_1 \vee T_2(t) = T_2 \vee T_1(t)$.

Let the r.i. spaces $(F, \|\cdot\|_F)$ and $(G, \|\cdot\|_G)$ over our measurable space have Tchebychev's characteristic functions correspondingly $T^{(F)}(t)$, $T^{(G)}(t)$. Let also a third space H be a (direct) sum of this spaces: $H = F + G$.

Theorem 5.1.

$$\max(T^{(F)}(t), T^{(G)}(t)) \leq T^{(H)}(t) \leq T^{(F)}(t) \vee T^{(G)}(t). \quad (5.1)$$

Proof. The left-hand side of bilateral inequality (5.1) is proved very simple. Let f_0 be a function (depending on the variable t) from the space F such that $\|f_0\|_F = 1$, $T_{f_0}(t) = T^{(F)}(t)$. Then we have for the function $h_0 = f_0 + 0 \in H$: $\|h_0\|_H = 1$ and $T_{h_0}(t) = T^{(F)}(t)$, therefore

$$T^{(H)}(t) \geq T^{(F)}(t)$$

and analogously

$$T^{(H)}(t) \geq T^{(G)}(t).$$

We will prove now the right-hand inequality in (5.1).

Let $h : \Omega \rightarrow R$ be any function from the space H with unit norm in this space. We can suppose without loss of generality by virtue of definition of sum of two spaces that exist two functions say $f, f \in F$ and $g, g \in G$ for which $h = f + g$ and

$$1 = \|h\|_H = \|f\|_F + \|g\|_G. \quad (5.2)$$

It follows from the equality (5.2) that

$$\|f\|F \leq 1, \|g\|G \leq 1$$

and therefore

$$T_f(t) \leq T^{(F)}(t), T_g(t) \leq T^{(G)}(t). \quad (5.3)$$

Let x be arbitrary number from the set $[0, 1]$ and $y = 1 - x$. We have:

$$T_h(t) \leq T_f(tx) + T_g(ty) \leq T^{(F)}(tx) + T^{(G)}(ty).$$

Since the value x is arbitrary in the closed interval $[0, 1]$, we conclude

$$T_h(t) \leq \inf_{x \in [0, 1]} [T^{(F)}(tx) + T^{(G)}(t(1 - x))] = [T^{(F)} \vee T^{(G)}](t). \quad (5.4)$$

Taking the supremum over $h : \|h\|H = 1$ we obtain

$$T^{(H)}(t) \leq [T^{(F)} \vee T^{(G)}](t). \quad (5.5)$$

This completes the proof of theorem 5.1.

Example 5.1. Let F, G be Orlicz's spaces over probabilistic space with diffuse measure and with the Young functions correspondingly

$$N_F(u) = |u|^{p_1} \log^{q_1}(e + |u|), N_G(u) = |u|^{p_2} \log^{q_2}(e + |u|),$$

$p_1, p_2 = \text{const} > 1, q_1, q_2 = \text{const}$. Then the space $H = F + G$ is also the Orlicz's space relative the Young function $N_h(u) = \max(N_F(u), N_G(u))$ and with the Tchebychev's function

$$T^{(H)}(t) \asymp \max(T^{(F)}(t), T^{(G)}(t)), t > 1.$$

6 Tchebychev's characteristic of Grand Lebesgue-Riesz spaces (GLS).

We recall first of all in this section for reader conventions some definitions and facts from the theory of GLS spaces.

Recently, see [2], [3], [4], [5], [6], [7], [9], [10], [11], etc. appears the so-called Grand Lebesgue Spaces $GLS = G(\psi) = G\psi = G(\psi; A, B)$, $A, B = \text{const}, A \geq 1, A < B \leq \infty$, spaces consisting on all the measurable functions $f : X \rightarrow R$ with finite norms

$$\|f\|G(\psi) \stackrel{\text{def}}{=} \sup_{p \in (A, B)} [\|f\|_p / \psi(p)]. \quad (6.1)$$

Here $\psi(\cdot)$ is some continuous positive on the *open* interval (A, B) function such that

$$\inf_{p \in (A, B)} \psi(p) > 0, \psi(p) = \infty, p \notin (A, B). \quad (6.2)$$

We will denote

$$\text{supp}(\psi) \stackrel{\text{def}}{=} (A, B) = \{p : \psi(p) < \infty, \}$$

The set of all ψ functions with support $\text{supp}(\psi) = (A, B)$ will be denoted by $\Psi(A, B)$.

These spaces are rearrangement invariant, see [1], and are used, for example, in the theory of probability [7], [10], [11]; theory of Partial Differential Equations [3], [6]; functional analysis [4], [5], [9], [11]; theory of Fourier series [10], theory of martingales [11], mathematical statistics [22], [23]; theory of approximation [15] etc.

Notice that in the case when $\psi(\cdot) \in \Psi(A, \infty)$ and a function $p \rightarrow p \cdot \log \psi(p)$ is convex, then the space $G\psi$ coincides with some *exponential* Orlicz space.

Conversely, if $B < \infty$, then the space $G\psi(A, B)$ does not coincide with the classical rearrangement invariant spaces: Orlicz, Lorentz, Marcinkiewicz etc.

Remark 6.1 If we introduce the *discontinuous* function

$$\psi_r(p) = 1, p = r; \psi_r(p) = \infty, p \neq r, p, r \in (A, B)$$

and define formally $C/\infty = 0$, $C = \text{const} \in R^1$, then the norm in the space $G(\psi_r)$ coincides with the L_r norm:

$$\|f\|_{G(\psi_r)} = \|f\|_r.$$

Thus, the Grand Lebesgue Spaces are direct generalization of the classical exponential Orlicz's spaces and Lebesgue spaces L_r .

Remark 6.2 The function $\psi(\cdot)$ may be generated as follows. Let $\xi = \xi(x)$ be some measurable function: $\xi : X \rightarrow R$ such that $\exists(A, B) : 1 \leq A < B \leq \infty, \forall p \in (A, B) |\xi|_p < \infty$. Then we can choose

$$\psi(p) = \psi_\xi(p) = |\xi|_p.$$

Analogously let $\xi(t, \cdot) = \xi(t, x), t \in T$, T is arbitrary set, be some *family* $F = \{\xi(t, \cdot)\}$ of the measurable functions: $\forall t \in T \xi(t, \cdot) : X \rightarrow R$ such that

$$\exists(A, B) : 1 \leq A < B \leq \infty, \sup_{t \in T} |\xi(t, \cdot)|_p < \infty.$$

Then we can choose

$$\psi(p) = \psi_F(p) = \sup_{t \in T} |\xi(t, \cdot)|_p.$$

The function $\psi_F(p)$ may be called as a *natural function* for the family F . This method was used in the probability theory, more exactly, in the theory of random fields, see [10].

More detail investigations of tail and fundamental functions of GLS see in [10], [11], [9].

We consider in this section only the cases $\mu = \mathbf{P}$ and $B < \infty$.

An important

Example 6.1. Let $B = \text{const} > 1$, $\beta = \text{const} > 0$ and let

$$\psi_{B,\beta}(p) = (B - p)^{-\beta}, \quad 1 \leq p < B \quad (6.3)$$

and $\psi_{B,\beta}(p) = \infty$ otherwise.

For instance: if $\Omega = (0, 1)$, $\mathbf{P} = m$ and $\xi_2(\omega) = \omega^{-1/2}$, then $\xi_2(\cdot) \in G\psi_{2,1/2}(\cdot)$.

Notice that for all positive values $\epsilon < 0.5$

$$\xi_2(\cdot) \notin G\psi_{2+\epsilon,1/2}(\cdot) \cup G\psi_{2,1/2-\epsilon}(\cdot)$$

and that the function $\psi_{2,1/2}(p)$ is equivalent to the natural function for the random variable $\xi_2(\cdot)$.

Lemma 6.1. Denote

$$\tilde{\psi} = p \cdot \log \psi(p), \quad p \in [1, B]. \quad (6.4)$$

Proposition:

$$\mathbf{A.} \quad T^{(G(\psi))}(t) \leq \exp \left(-\tilde{\psi}^*(\log t) \right), \quad t > 2. \quad (6.5)$$

where $h^*(\cdot)$ denotes the classical Young-Fenchel, or Legendre transform:

$$h^*(x) = \sup_y (xy - h(y)).$$

B. For the spaces $G\psi_{B,\beta}(\cdot)$ it true also the converse inequality up to dilation:

$$T^{(G(\psi_{B,\beta}))}(t) \geq \exp \left(-\tilde{\psi}^*(\log t / C(B, \beta)) \right), \quad t > 2. \quad (6.6)$$

Proof. **A.** Let $\|\xi\|_{G\psi} = 1$; then $\|\xi\|_p \leq \psi(p)$, $\mathbf{E}|\xi|^p \leq \psi^p(p)$. We obtain using the Tchebychev's inequality:

$$T_\xi(t) \leq \exp \left((p \log t - p \log \psi(p)) \right).$$

The assertion (6.5) it follows after an optimization over p .

The proposition (6.6) is proved in the article [11]; see also [9].

Example 6.2. Denote $\psi_m(p) = p^{1/m}$, $1 \leq p < \infty$, $m = \text{const} > 0$. Proposition:

$$\xi \in G\psi_m, \quad \xi \neq 0 \Leftrightarrow \exists C = \text{const} \in (0, \infty), \quad T_\xi(t) \leq \exp(-C t^m).$$

We will formulate the main result of this section, which may be obtained after simple calculations basing on the lemma 6.1.

Theorem 6.1. There exists a *non-regular* r.i. space over the probabilistic space with diffuse measure, namely the space $G\psi_{B,\beta}$ with $B > 1$, $\beta > 0$.

Proof. Let us consider the space $G\psi_{B,\beta}$. In detail:

$$T^{(G\psi_{B,\beta})}(t) \asymp t^{-B} (\log t)^{\beta B}, \quad t \rightarrow \infty,$$

$$\phi_{G\psi_{B,\beta}}(\delta) \asymp \delta^{1/B} |\log \delta|^\beta, \quad \delta \rightarrow 0+,$$

so that at $t \rightarrow \infty$

$$\left[\frac{1}{\phi_{G\psi_{B,\beta}}(1/t)} \right]^{-1} \asymp t^B (\log t)^{\beta B},$$

$$\frac{1}{T^{(G\psi_{B,\beta})}(t)} \asymp t^B (\log t)^{-\beta B}.$$

References

- [1] Bennet C. and Sharpley R. Interpolation of operators. Orlando, Academic Press Inc., 1988.
- [2] CAPONE C., FIORENZA A., KRBEC M. On the Extrapolation Blowups in the L_p Scale. *Collectanea Mathematica*, **48**, 2, (1998), 71 - 88.
- [3] FIORENZA A. Duality and reflexivity in grand Lebesgue spaces. *Collectanea Mathematica* (electronic version), **51**, 2, (2000), 131 - 148.
- [4] FIORENZA A., AND KARADZHOV G.E. Grand and small Lebesgue spaces and their analogs. Consiglio Nazionale Delle Ricerche, Istituto per le Applicazioni del Calcolo Mauro Picone, Sezione di Napoli, Rapporto tecnico n. 272/03, (2005).
- [5] IWANIEC T., AND SBORDONE C. On the integrability of the Jacobian under minimal hypotheses. *Arch. Rat.Mech. Anal.*, 119, (1992), 129 - 143.
- [6] IWANIEC T., P. KOSKELA P., AND ONNINEN J. Mapping of finite distortion: Monotonicity and Continuity. *Invent. Math.* 144 (2001), 507 - 531.
- [7] KOZACHENKO YU. V., OSTROVSKY E.I. (1985). The Banach Spaces of random Variables of subgaussian type. *Theory of Probab. and Math. Stat.* (in Russian). Kiev, KSU, **32**, 43 - 57.
- [8] Krein S.G., Petunin Yu.V. and Semenov E.M. Interpolation of Linear operators. New York, AMS, 1982.
- [9] LIFLYAND E., OSTROVSKY E., SIROTA L. Structural Properties of Bilateral Grand Lebesgue Spaces. *Turk. J. Math.*; **34** (2010), 207-219.
- [10] OSTROVSKY E., SIROTA L. Universal adaptive estimations and confidence intervals in the non-parametrical statistics. Electronic Publications, arXiv.mathPR/0406535 v1 25 Jun 2004.
- [11] OSTROVSKY E., ZELIKOV YU. Adaptive Optimal Nonparametric Regression and Density Estimation based on Fourier - Legendre Expansion. Electronic Publication, arXiv:0706.0881v1 [math.ST] 6 Jun 2007.

- [12] OSTROVSKY E., SIROTA L. Optimal adaptive nonparametric denoising of multidimensional-time signal. Electronic Publication, arXiv:0809.3021v1 [physics.data-an] 17 Sep 2008.
- [13] OSTROVSKY E.I. Exponential Estimations for Random Fields. Moscow - Obninsk, OINPE, (1999), in Russian.
- [14] OSTROVSKY E., ROGOVER E. AND SIROTA L. Optimal Adaptive Signal Detection and Measurement. In: Abstracts of the International Symposium on STOCHASTIC MODELS IN RELIABILITY ENGINEERING, LIFE SCIENCES AND OPERATIONS MANAGEMENT, Beer Sheva, Israel, (2010), p. 175.
- [15] OSTROVSKY E., SIROTA L. Nikol'skii-type inequalities for rearrangement invariant spaces. arXiv:0804.2311v1 [math.FA] 15 Apr 2008.
- [16] OSTROVSKY E. AND SIROTA L. Moment Banach spaces: theory and applications. HIAT Journal of Science and Engineering, C, Volume 4, Issues 1 - 2, pp. 233 - 262, (2007).
- [17] OSTROVSKY E., AND SIROTA L. Boundedness of operators in bilateral Grand Lebesgue spaces, with exact and weakly exact constant calculation. arXiv:1103.2963 [math.FA] 15 Apr 2011.
- [18] OSTROVSKY E., SIROTA L., ROGOVER E. Integral Operators in bilateral Grand Lebesgue Spaces. arXiv:0912.2538 [math.FA] 13 Dez 2009.
- [19] OSTROVSKY E., AND SIROTA L. Tail estimates for martingale under "LLN" norming sequense. arXiv:1207.1908v1 [math.PR] 8 Jul 2008.
- [20] OSTROVSKY E., AND SIROTA L. Monte-Carlo method for multiple parametric integrals calculation and solving of linear integral Fredholm equations of a second kind, with confidence regions in uniform norm. arXiv:1101.5381 v1 [math.FA] 27 Jan 2011
- [21] OSTROVSKY E., AND SIROTA L. Non-improved uniform tail estimates for mormed sums of independent random variables with heavy tails, with applications. arXiv:1110.4879 v1 [math.PR] 21 Oct 2011
- [22] OSTROVSKY E., AND SIROTA L. Adaptive multidimensional-time spectral Measurements in technical diagnosis. Communications in dependability and Managements (CDQM), Vol. 9, No 1, (2006), pp. 45-50.
- [23] OSTROVSKY E., SIROTA L. Adaptive optimal measurements in the technical diagnostics, reliability theory and information theory. Procedings 5th international conference on the improvement of the quality, reliability and long usage of technical systems and technological processes, (2006), Sharm el Sheikh, Egypt, p. 65-68.